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Influence of Instantaneous Fertility Decline to Replacement Level on Population Growth : An Alternative Model

1. The Problem

IN a very interesting article, Keyfitz (1971) has shown that "if age-specific birth rates drop immediately to the level of bare replacement the ultimate stationary number of a population will be given by

$$I = \frac{be_0^0}{r\mu}\left(\frac{R_0 - 1}{R_0}\right) \qquad (1)$$

multiplied by the present number". In (1), b and r are the intrinsic birth rate and the intrinsic rate of growth, R_a the net reproduction rate, e[j the expectation of life at birth and M the average age of child bearing in the resulting stationary population. This result is obtained by assuming that the population is stable and the regime of age-specific birth rates m_x transforms immediately to m_x/R_0 where m_x refers to one-sex model only. Keyfitz has acknowledged that "this is not the only way an NRR can be constituted, and not even the most probable way" and that "the fall in the birth rate is likely to be more rapid for older women than for younger".

The purpose of this paper is to examine the modifications in (1) resulting from a pattern of fall in the birth rate that is more rapid for older women by using a variable multiplier to m_x. The use of e^{-rx} immediately suggests itself, in which r is the intrinsic rate of growth corresponding to m_x and survivorship function l_x. In other words, r is the real root of the integral equation

$$\int_0^\infty e^{-rx} l_x m_x dx = 1. \tag{2}$$

It may be pointed out that Keyfitz has shown with numerical examples and population projections that equation (1) fails when the initial population is not approximately stable. This should also be expected from the present model which is based merely on an alternative pattern of fertility decline in a stable population. However, for approximately stable populations, Keyfitz found that the results were close to the model values, as they should be. To avoid unnecessary duplications, such projections will not be undertaken here since the relative age compositions of the ultimate stationary populations will be identical regardless of the differences in the schedules of age-specific fertility rates, as long as the mortality rates remain the same. The ultimate populations will of course differ with respect to their sizes, the magnitudes of which can be ascertained mathematically, as has been demonstrated in the paper.

The question regarding the actual pattern of instantaneous fertility decline, is still at a theoretical level and cannot be empirically answered. However, for the four countries used as examples in the present paper, the fertility declines during recent years tend to confirm the pattern anticipated by Keyfitz. That is to say, the decline is indeed more rapid for older women and an exponential curve provides a simple and reasonable approximation of this decline. In addition, it may be mentioned that although, the exponential pattern of decline has been suggested in this paper as an alternative, the algebraic solution has also been worked out for the general case (Section 5, equation 30), in which no assumption is made about the pattern of fertility except however, that of an instantaneous decline to a level of stationarity.

What follows next is an analysis of the impact of an exponential decline on population growth and a comparison of results with the model developed by Keyfitz. Particular cases, corresponding to varying patterns of fertility rates have also been examined (Section 3) and results have been obtained for four countries (Section 4).

2. Functional Expression for the Stationary Population

It is well known that the number of births $B(t)$ at a given time t, as the population approaches stability, is approximated by Qe^{rt} where

$$Q = \frac{\int_0^\beta e^{-rt} G(t)dt}{\int_\alpha^\beta xe^{-rx} l_x m_x dx} \tag{3}$$

where $G(t)$ are the births to women who were present at the start of the process and (α, β) is the reproductive interval. If l_x remains unchanged and m_x is transformed to

$$m_x^* = e^{-rx} m_x \tag{4}$$

the rate of growth intrinsic to l_x and m_x^* is zero and the corresponding Q simplifies to

$$Q_0 = \frac{\int_0^\beta G(t)dt}{\int_\alpha^\beta x l_x m_x^* dx} \tag{5}$$

which also

$$= \frac{\int_0^\beta G(t)dt}{\int_\alpha^\beta xe^{-rx} l_x m_x dx} \tag{6}$$

because of (4). Interestingly enough, the denominator of (6) is the average age of childbearing μ not only in the resulting stationary population but also in the stable population generated by l_x and m_x.

The numerator of Q_0 depends on $G(t)$ and the latter can be expressed as

$$G(t) = \int_{\alpha-t}^{\beta-t} P_x \frac{l_{x+t}}{l_x} m_{x+t}^* dx \tag{7}$$

where P_x is the population aged x at the beginning of the process. Following

Keyfitz, if B is the number of births at the beginning of fertility decline and P_x is stable, (7) simplifies to

$$B \int_{\alpha-t}^{\beta-t} e^{-r(2x+t)} l_{x+t} m_{x-t} dx \tag{8}$$

because of (4). The numerator of Q_0 can be written as a double integral

$$\int_0^{\beta} G(t)dt = \int_0^{\infty} \int_0^{\infty} e^{-r(2x+t)} l_{x+t} m_{x+t} dt dx \tag{9}$$

where the order of integration has been interchanged and the independent limits of integration extended to (0, ∞).

Writing
$$x + t = a$$
$$x = x$$

so that $x \leqslant a \leqslant \infty$ and $dtdx = dadx$, (9) can be written as

$$\int_0^{\infty} \int_x^{\infty} e^{-rx} e^{-ra} l_a m_a da$$

or as

$$\int_0^{\infty} \int_0^{\infty} e^{-rx} e^{-ra} l_a m_a da - \int_0^{\infty} \int_0^{x} e^{-rx} e^{-ra} l_a m_a da. \tag{10}$$

The first integral of (10) can be written as the product of

$$\int_0^{\infty} e^{-rx} dx \quad \text{and} \quad \int_0^{\infty} e^{-ra} l_a m_a da$$

which are 1/r and 1 respectively. The second integral can be simplified by first writing

$$F(x) = \int_0^{x} e^{-ra} l_a m_a da$$

so that

$$\frac{dF(x)}{dx} = e^{-rx} l_x m_x,$$

and then integrating by parts,

$$\int_0^\infty e^{-rx}F(x)dx$$

reduces to

$$\frac{1}{r}\int_0^\infty e^{-2rx}l_x m_x dx. \tag{11}$$

Substitution of all these in (6) produces

$$Q_0 = \frac{B}{r\mu}\left(1 - \int_0^\infty e^{-2rx}l_x m_x dx\right). \tag{12}$$

Since the size of the stationary population can be expressed as a product of the births and the life expectancy, this size relative to that at the beginning of the process, say P, can be expressed as

$$I^* = \frac{be_0^0}{r\mu}\left(1 - \int_0^\infty e^{-2rx}l_x m_x dx\right) \tag{13}$$

which can be compared with (1). The major difference is that $1/R_0$ or $1/\int_0^\infty l_x m_x dx$ in the latter has been replaced by $\int_0^\infty e^{-2rx}l_x m_x dx$ in the former. Interestingly enough, the latter is greater than or equal to the former, because

$$\int_0^\infty e^{-2rx}l_x m_x dx \int_0^\infty l_x m_x dx \geqslant \left(\int_0^\infty e^{-rx}l_x m_x dx\right)^2 \tag{14}$$

according to Cauchy-Schwarz inequality and the fact that the right-hand side equals 1.

Similarly, it can be shown that the average age of childbearing in the resulting stationary population with age specific birth rate as $e^{-rx}m_x$ is smaller than or equal to that when the age specific birth rate is given by $m(x)/R$. This is so because the derivative with respect to u of

$$k(u) = \int_0^\infty xe^{-ux} l_x m_x dx \Big/ \int_0^\infty e^{-ux} l_x m_x dx \tag{15}$$

is

$$-\frac{\int_0^\infty x^2 e^{-ux} l_x m_x dx \int_0^\infty e^{-ux} l_x m_x dx - \left(\int_0^\infty xe^{-ux} l_x m_x dx \right)^2}{\left(\int_0^\infty e^{-ux} l_x m_x dx \right)^2} \tag{16}$$

which is negative because of Cauchy-Schwartz inequality. Therefore, the value of (15) for $u = 0$ is greater than that for $u = r$, where $r > 0$.

Taking both of these inequality relationships into consideration while comparing the index values obtained by the two models, it becomes apparent that both the numerator and the denominator of (1) are larger than the corresponding quantities of (13). The net effect of this relationship on the index is not readily apparent and may have to be analyzed in individual cases.

3. Simplifications Based on Varying Assumptions About the Distribution of Net Maternity Function

It may be pointed out that the resolution of (13) in terms of a first few moments is not possible without additional assumptions about the functional forms of the net maternity functions. Different assumptions about its form have been made in the past, three of which are examined in the following.

a. Normality Assumption of Lotka (1939)

The net maternity function can be expressed as

$$\frac{R_0}{\sigma\sqrt{2\pi}} e^{-(x-\mu)^2/2\sigma^2} \tag{17}$$

where μ is the average age of childbearing and σ^2 the variance. The integral

$$\int_0^\infty e^{-2rx} l_x m_x dx$$

can be written as

$$\frac{R_0}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-2rx} e^{-(x-\mu)^2/2\sigma^2} dx = R_0\, e^{-2r\mu+2\sigma^2 r^2} \qquad (18)$$

which follows from the well known expression of the moment generating function. Using the approximate equality relationship

$$R_0 = e^{r\mu - \frac{1}{2}\sigma^2 r^2}$$

(18) can be further simplified as

$$\int_0^\infty e^{-2rx} l_x m_x dx = \frac{e^{r^2\sigma^2}}{R_0}. \qquad (20)$$

Substitution of (18) or (20) in (13) will produce an estimate of I^*.

b. Wicksell's Incomplete Gamma Function (1931)

According to Wicksell, $l_x m_x dx$ can be replaced by

$$\frac{Kc^k x^{k-1} e^{-cx}}{\Gamma(k)}\, dx;\ x \geqslant 0. \qquad (21)$$

A modified version of the above was suggested by Mitra (1970) as

$$\frac{Kc^k (x-\alpha)^{k-1} e^{-cx} dx}{\Gamma(k)};\ x \geqslant \alpha \qquad (22)$$

where a is the lower limit of the childbearing period. It is easy to see that the integral

$$\int_0^\infty e^{-2rx} l_x m_x dx = e^{-r\alpha}\left(\frac{r+c}{2r+c}\right)^k.$$

For such a distribution

$$R_0 = Ke^{-cx}$$

$$c = \mu_1'/\sigma^2$$

$$k = \mu_1'^2/\sigma^2$$

where μ_1' is obtained by using α as the arbitrary origin, so that the average age of childbearing μ is equal to $\mu_1' + \alpha$.

c. Pearson's Type 1 Distribution

Mitra (1970) observed that among several approximations the best result is obtained when the net maternity function is expressed in the form of Pearson's type 1 distribution. Accordingly, using the limits α and β, one can write

$$l_x m_x dx = \frac{R_0(x - \alpha)^{m_1-1}(\beta - x)^{m_2-1}}{(\beta - \alpha)^{m_1+m_2-1}B(m_1, m_2)} dx; \quad \alpha \leqslant x \leqslant \beta \tag{23}$$

where B stands for the Beta function conveniently defined as

$$B(m_1, m_2) = \int_0^1 x^{m_1-1}(1 - x)^{m_2-1}dx; \quad m_1, m_2 > 0. \tag{24}$$

It can be seen that the integral

$$\int_0^\infty e^{-2rx} l_x m_x dx$$

can then be written as

$$\frac{R_0 e^{-2r\alpha}}{(B - \alpha)^{m_1+m_2-1}B(m_1, m_2)} \sum_{i=0}^{\infty} \frac{(-2r)^i}{i!} \int_\alpha^\beta (x - \alpha)^{m_1+i-1}(\beta - x)^{m_2-1}dx$$

and simplified into

$$\frac{R_0 e^{-2r\alpha}}{B(m_1, m_2)} \sum_{i=0}^{\infty} \frac{(-2r)^i}{i!} (\beta - \alpha)^i B(m_1 + i, m_2). \tag{25}$$

Noting the relation between Beta and Gamma functions as

$$B(m_1, m_2) = \Gamma(m_1)\Gamma(m_2)/\Gamma(m_1 + m_2)$$

(25) can be written as

$$R_0 e^{-2r\alpha}\left[1 + \sum_{i=1}^{\infty} \frac{(-t)^i}{i!} \frac{m_1(m_1+1)\cdots(m_1+i-1)}{(m_1+m_2)(m_1+m_2+1)\cdots(m_1+m_2+i-1)}\right] \tag{26}$$

in which $t = 2r(\beta - \alpha)$. It can be shown that since $B(m_1, m_2) = B(m_2, m_1)$, another expression for the integral can be obtained as

$$R_0 e^{-2r}\left[1 + \sum \frac{(t)^i}{i!} \frac{m_2(m_2+1)\cdots(m_2+i-1)}{(m_1+m_2)(m_1+m_2+1)\cdots(m_1+m_2+i-1)}\right]. \tag{27}$$

However, for empirical evaluation of the results, the alternating series in (26) is preferable because of its rapid convergence due to the fact that generally, $m_1 < m_2$.

4. Results

Of the three models representing the net maternity function, the normal is the simplest as for a given value of R_0, only two more parameters must be known or estimated for purposes of graduation, in contrast with three for the type III and four for the type I distributions. However, for reasons of simplicity, the number of parameters in the latter distributions can be reduced to two by assuming say, $\alpha = 15$ for the type III and in addition $\beta = 45$ for the type I distributions. The parameters necessary to estimate I^* are thereby reduced to b, r, e_0^0, the average and the variance of the distribution of the net maternity function. While comparing the index values with those presented by Keyfitz (1971, Table 3), it was observed that the latter two parameters were not needed and therefore were not presented by him. However, it was found that average age of net maternity can be approximately obtained for small r from (16), when the same is known for the stationary population in which the age specific birth rate is m_x/R_0. For $u = r$, the formula reduces to

$$\left[\frac{dk(u)}{du}\right]_{u=r} = -\sigma^2 \tag{28}$$

where a^2 is the variance of the distribution of net maternity function. Thus

$$\mu^* = \mu - r\sigma^2 \qquad (29)$$

where μ^* and μ are the average ages corresponding to m_X^* and m_x/R_0 respectively. All these parameters are shown in Table 1, in which the index values obtained by Keyfitz *(ibid.)* are reproduced in the row corresponding to the variable I (see equation 1). The I* values are obtained from the model developed in this paper and subjected to three different patterns assumed for the age specific fertility rates.

TABLE 1—RELATIVE SIZE OF ULTIMATE STATIONARY POPULATION I AND A FEW OTHER IMPORTANT PARAMETERS, ASSUMING IMMEDIATE DECLINE IN FERTILITY TO REPLACEMENT LEVEL FROM A STABLE STATE

Country *Parameters and I values*	*Chile 1965*	*Colombia 1965*	*Italy 1966*	*U.S. 1967*
(1)	*(2)*	*(3)*	*(4)*	*(5)*
1000b	33.4	38.8	16.7	17.8
1000 r	23.5	29.0	5.58	7.38
R_o	1.95	2.31	1.17	1.21
e_0^0	62.7	61.7	74.2	74.2
μ	29.1	29.6	28.6	26.3
μ^*	27.9	28.2	28.4	26.0
μ^2†	49.4	50.9	37.4	34.9
I	1.49	1.59	1.13	1.18
I* (normal)	1.52	1.61	1.16	1.21
I* (Type III)	1.49	1.57	1.08	1.23
I* (Type I)	1.41	1.48	1.13	1.19

† I am grateful to Professor Nathan Keyfitz for providing me with these variances.

5. Concluding Observations

Four sets of index values presented in Table 1 are quite comparable as they are expected to be. Of the four estimates the size of the stationary population relative to that of the stable population at the beginning of the instantaneous fertility decline is lowest according to type I assumptions of net maternity function for countries experiencing rapid rate of growth. For these two countries, namely, Chile and Colombia differences in ultimate population sizes are of the order of six to seven percent which is quite significant in terms of absolute numbers. However, fertility decline can rarely be expected to be instantaneous and the ultimate size of the stationary population, if and whenever attained, will be larger than any of the estimates obtained from the index values. Such estimates when fertility decline is spread over a number of years, can be made by making appropriate adjustments along the lines suggested by Keyfitz.

One final observation may be made at this point. If the initial population is stable and the fertility decline to replacement level is based on a set of wj values different from m_x/R_0 or from $m_x e^{-rx}$ as assumed respectively by Keyfitz and by this investigator, the mathematical expression for the index value simplifies to

$$I^* = \frac{be_0^0}{r\mu^*}\left(1 - \int_0^\infty e^{-rx} l_x m_x^* dx\right) \tag{30}$$

where

$$\int_0^\infty l_x m_x^* dx = 1.$$

Phis can easily be verified by following steps similar to those described in lection 2, that were used to develop (8)-(13). The algebraic solution of (30) s given by

$$I^* = be_0^0\left[1 - \frac{r}{2\mu^*}(\mu_2^* + \mu^{*2}) + \cdots\right]$$

obtained by the expansion of the exponential in (30) and subsequent simplifications. The series will usually converge, however, knowledge of the moments beyond the second of the stationary population may be required for the solution. In practice, however, μ_2^* will be small compared to $(\mu^*)^2$ and as a trial solution one may use $be_0^0\left(1 - \frac{r\mu^*}{2}\right)$ where for μ^*, the average

age at childbearing of the stable population may be substituted. For Chile, Colombia, Italy and U.S. values of I* thus obtained are 1.41, 1.42, 1.14 and 1.19 respectively which compared with others, appear to be quite satisfactory.

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REFERENCES

1. Keyfitz, N., 1968, *Introduction to the Mathematics of Population,* Reading, Massachusetts, Addison-Wesley.
2. ___, 1971, On the momentum of population growth, *Demography,* 8(1), 71-80.
3. Lotka, A. J., 1939, Theorie analytique des associations biologiques. Part II. Analyse demographique avec application particuliere a 1'espece humaine. (Actualities Scientifiques et Industrilles, No. 780.) Paris : Hermann and Cie.
4. Mitra, S., 1967, The pattern of age-specific fertility rates, *Demography,* 4(2), 894-906.
5. Wicksell, S. D., 1931, Nuptiality, fertility and reproductivity, *Skandinavisk Aktuarietidskrift,* **14,** 125-157.